Discrete Applied Mathematics 159 (2011) 311–321

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

# On exact blockers and anti-blockers, $\Delta$-conjecture, and related problems

Vladimir Gurvich

*RUTCOR, Rutgers University, 640 Bartholomew Road, Piscataway NJ 08854-8003, United States*

ARTICLE INFO

*Article history:*
Received 11 August 2009
Received in revised form 13 November 2010
Accepted 19 November 2010
Available online 16 December 2010

*Keywords:*
Blocker
Anti-blocker
Exact blocker
Exact anti-blocker
Read-once Boolean function
CIS graph
CIS $d$-graphs
$\Delta$-conjecture
Box-partition
Solid box-partition

ABSTRACT

Let us consider two binary systems of inequalities (i) $Cx \geq e$ and (ii) $Cx \leq e$, where $C \in \{0, 1\}^{m\times n}$ is an $m \times n(0, 1)$-matrix, $x \in \{0, 1\}^n$, and $e$ is the vector of $m$ ones. The set of all support-minimal (respectively, support-maximal) solutions $x$ to (i) (respectively, to (ii)) is called the *blocker* (respectively, *anti-blocker*).

A blocker $\mathcal{B}$ (respectively, anti-blocker $\mathcal{A}$) is called *exact* if $Cx = e$ for every $x \in \mathcal{B}$ (respectively, $x \in \mathcal{A}$).

Exact blockers can be completely characterized. There is a one-to-one correspondence between them and $P_4$-free graphs (along with a well-known one-to-one correspondence between the latter and the so-called read-once Boolean functions). However, the class of exact anti-blockers is wider and more sophisticated. We demonstrate that it is closely related to the so-called CIS graphs, more general $\ell$-CIS $d$-graphs, and $\Delta$-conjecture.



## 1. Graphs and hypergraphs; basic definitions

A hypergraph $\mathcal{H} \subseteq 2^V$ on the vertex-set $V = V(\mathcal{H}) = \{v_1, \ldots, v_n\}$ is a non-empty family of non-empty subsets $H \subseteq V$ called its *edges*, that is, $H \in \mathcal{H}$. For convenience, we will assume from now on that every vertex belongs to an edge, or in other words, that $V = \cup_{H\in\mathcal{H}} H$.

A subset $A \subseteq V$ is called an *independent (or stable) set* of $\mathcal{H}$ if $A$ contains no edge, that is, if $H \subseteq A$ for no $H \in \mathcal{H}$. An independent set $A$ is called *maximal* if none of its proper supersets $A' \supset A$ is independent, that is, if $A' \supseteq H$ for some $H \in \mathcal{H}$.

A subset $B \subseteq V$ is called *transversal* to $\mathcal{H}$ if $B$ meets all edges of $\mathcal{H}$, that is, if $B \cap H \neq \emptyset$ for every $H \in \mathcal{H}$. A transversal $B$ is called *minimal* if none of its proper subsets $B' \subset B$ is a transversal, that is, if $B' \cap H = \emptyset$ for some $H \in \mathcal{H}$.

Obviously, the complement to a (minimal) transversal is an (maximal) independent set and vice versa.

A hypergraph $\mathcal{H} \subseteq 2^V$ is called a *graph* if each of its edges $H \in \mathcal{H}$ consists of precisely two vertices; such vertices are called *adjacent*. Standardly, we denote a graph by $G$ (rather than by $\mathcal{H}$) and the set of its edges by $E = E(G)$. The complementary graph $\overline{G}$ of $G$ is defined by the same vertex-set, $V(\overline{G}) = V(G)$, and the complementary edge-set, $(v', v'') \in E(\overline{G})$ if and only if $(v', v'') \notin E(G)$ for any two distinct $v', v'' \in V(G)$.

A set of pairwise adjacent (respectively, non-adjacent) vertices of a graph $G$ is called a *clique* (respectively, an independent or stable set) of $G$. Obviously, an (maximal) independent set of $G$ is a (maximal) clique in $\overline{G}$ and vice versa.

To each hypergraph $\mathcal{H} \subseteq 2^V$ let us assign its *co-occurrence graph* $G = G(\mathcal{H})$ on the same vertex-set $V = V(G) = V(\mathcal{H})$ and such that two vertices $v', v'' \in V$ are adjacent in $G$ if and only if they are distinct, $v' \neq v''$, and adjacent in $\mathcal{H}$, that is, $v', v'' \in H$ for an edge $H \in \mathcal{H}$.

*E-mail address:* gurvich@rutcor.rutgers.edu.

0166-218X/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.dam.2010.11.014

**Example 1.** Distinct hypergraphs can have the same co-occurrence graph. Consider three examples:

$$\mathcal{H}_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\} \quad \text{and} \quad \mathcal{H}_2 = \{(v_1, v_2, v_3)\}$$

both correspond to the complete graph on the ground set $V = \{v_1, v_2, v_3\}$;

$$\mathcal{H}_3 = \{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_1)\} \quad \text{and} \quad \mathcal{H}_4 = \mathcal{H}_3 \cup \{(v_1, v_3, v_5)\}$$

both generate the same graph, so-called sun or 3-anti-comb; finally

$$\mathcal{H}_5 = \{(v_1, v_3, v_6), (v_1, v_4, v_5), (v_1, v_4, v_6), (v_2, v_3, v_5), (v_2, v_3, v_6), (v_2, v_4, v_5)\} \quad \text{and}$$
$$\mathcal{H}_6 = \mathcal{H}_5 \cup \{(v_1, v_3, v_5)\} \cup \{(v_2, v_4, v_6)\}$$

both generate the same complete 3-partite graph of size $2 \times 2 \times 2$.

Conversely, with a graph $G$ let us associate its *clique hypergraph* $\mathcal{H}_C = \mathcal{H}_C(G)$ and its *stable-set hypergraph* $\mathcal{H}_S = \mathcal{H}_S(G)$ as follows: both have the same vertex-set, $V = V(G) = V(\mathcal{H}_C) = V(\mathcal{H}_S)$, while the edges are all maximal cliques and all maximal stable sets of $G$, respectively.

A hypergraph $\mathcal{H}$ will be called *completely clique-maximal* if it is the clique hypergraph of its own co-occurrence graph, that is, $\mathcal{H} = \mathcal{H}_C(G(\mathcal{H}))$. Let us remark that completely clique-maximal hypergraphs are also called *normal* in the literature; see, for example, [18,17,16].

In the above example, $\mathcal{H}_2$, $\mathcal{H}_4$, and $\mathcal{H}_6$ are completely clique-maximal, while $\mathcal{H}_1$, $\mathcal{H}_3$, and $\mathcal{H}_5$ are not.

By definition, for any hypergraph $\mathcal{H}'$ there is a unique completely clique-maximal hypergraph $\mathcal{H}$ with the same co-occurrence graph, $G = G(\mathcal{H}) = G(\mathcal{H}')$; obviously, $\mathcal{H} = \mathcal{H}_C(G(\mathcal{H}'))$.

Furthermore, a hypergraph $\mathcal{H}$ will be called *clique-maximal* if $\mathcal{H} \subseteq \mathcal{H}_C(G(\mathcal{H}))$, or in other words, if each edge of $\mathcal{H}$ is a *maximal* clique in $G(\mathcal{H})$. Yet, some maximal cliques of $G(\mathcal{H})$ might be missing in $\mathcal{H}$.

In the above example, all hypergraphs are clique-maximal, except for $\mathcal{H}_1$.

Finally, let us recall that $\mathcal{H}$ is called a *Sperner* hypergraph if none of its edges contains another one, that is, $H' \subseteq H''$ for no distinct $H', H'' \in \mathcal{H}$. Obviously, all six hypergraphs of Example 1 are Sperner ones.

It is easily seen that, in general, the above three families, of
(i) completely clique-maximal, (ii) clique-maximal, and (iii) Sperner hypergraphs, are nested,
(i) $\subset$ (ii) $\subset$ (iii). Example 1 shows that both containments are strict.

Given a hypergraph $\mathcal{H}$ with $n$ vertices, $V(\mathcal{H}) = \{v_1, \dots, v_n\}$, and $m$ edges, $\mathcal{H} = \{H_1, \dots, H_m\}$, its *incidence* matrix $C = C(\mathcal{H})$ is defined as an $m \times n$ (0, 1)-matrix whose entry $c(i, j)$ is 1 whenever $v_i \in H_j$ and 0 otherwise. We refer the reader to the monograph [4], by Claude Berge, for more concepts and details.

## 2. Blockers and anti-blockers

The hypergraph $\mathcal{B} = \mathcal{B}(\mathcal{H})$ of all minimal transversals to $\mathcal{H}$ is called the *blocker* of $\mathcal{H}$.

By definition, $\mathcal{B}$ is a Sperner hypergraph and $\cup_{B \in \mathcal{B}} B = V(\mathcal{B}) \subseteq V$.

If $\mathcal{H}$ is a Sperner hypergraph too then it is obvious and well known that
(i) $V(\mathcal{B}) = \cup_{B \in \mathcal{B}} B = \cup_{H \in \mathcal{H}} H = V(\mathcal{H})$ and (ii) $\mathcal{H}$ is the blocker of $\mathcal{B}$.

In this case, hypergraphs $\mathcal{H}$ and $\mathcal{B} = \mathcal{B}(\mathcal{H})$ are called dual and notation $\mathcal{B} = \mathcal{H}^d$ or, equivalently, $\mathcal{B}^d = \mathcal{H}$ is used. In other words, mapping $\mathcal{B}$ is an involution, that is, $\mathcal{B}(\mathcal{B}(\mathcal{H})) \equiv \mathcal{H}$ for any Sperner hypergraph $\mathcal{H}$.

In general, an arbitrary (not necessarily Sperner) hypergraph $\mathcal{H}$ can be reduced to a Sperner hypergraph $\mathcal{H}'$ by successive elimination of every edge that contains another edge. It is clear that $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}')$.

Given a hypergraph $\mathcal{H} \subseteq 2^V$, a subset $A \subseteq V$ is called *anti-blocking* if $A$ meets each edge of $\mathcal{H}$ in at most one vertex, that is, if $|A \cap H| \le 1$ for all $H \in \mathcal{H}$. Standardly, an anti-blocking set $A$ is called *maximal* if none of its proper superset is anti-blocking, that is, if $|A' \cap H| \ge 2$ for some $H \in \mathcal{H}$ whenever $A' \supset A$.

The hypergraph $\mathcal{A} = \mathcal{A}(\mathcal{H})$ of all maximal anti-blocking sets of $\mathcal{H}$ is called the *anti-blocker* of $\mathcal{H}$.

By definition, $\mathcal{A}(\mathcal{H})$ is a Sperner hypergraph and $\cup_{A \in \mathcal{A}} A = V(\mathcal{A}) = V$. It is also not difficult to verify that $\mathcal{A}(\mathcal{H}_C(G)) = \mathcal{H}_S(G)$. More generally, $\mathcal{A}(\mathcal{H}) = \mathcal{H}_S(G(\mathcal{H}))$ for each hypergraph $\mathcal{H}$, Sperner or not.

In particular, the anti-blocker $\mathcal{A}(\mathcal{H})$ depends only on the co-occurrence graph of $\mathcal{H}$. In other words, all hypergraphs with the same co-occurrence graph have the same anti-blocker. Of course, by symmetry, $\mathcal{A}(\mathcal{H}_S(G)) = \mathcal{H}_C(G)$ for any graph $G$.

In general, $\mathcal{A}(\mathcal{A}(\mathcal{H})) = \mathcal{H}$ if and only if $\mathcal{H}$ is completely clique-maximal, or in other words, if $\mathcal{H} = \mathcal{H}_C(G)$ (or equivalently, $\mathcal{H} = \mathcal{H}_S(G)$) for a graph $G$.

Even more generally (but still obviously) $\mathcal{A}(\mathcal{A}(\mathcal{H})) = H_C(G(\mathcal{H}))$ for all $\mathcal{H}$. In general, for an arbitrary (not necessarily clique-maximal or Sperner) hypergraph $\mathcal{H}'$ let us consider the corresponding completely clique-maximal hypergraph $\mathcal{H} = \mathcal{H}_C(G(\mathcal{H}'))$. It is clear that $G(\mathcal{H}) = G(\mathcal{H}')$ and $\mathcal{A}(\mathcal{H}) = \mathcal{A}(\mathcal{H}')$.

It is easily seen that blocker $\mathcal{B}(\mathcal{H})$ (respectively, anti-blocker $\mathcal{A}(\mathcal{H})$) can be equivalently redefined as the set of all support-minimal (respectively, support-maximal) binary solutions $x \in \{0, 1\}^n$ of the binary system $Cx \ge e$ (respectively, $Cx \le e$), where $C = C(\mathcal{H})$ is the $m \times n$ incidence matrix of $\mathcal{H}$ and $e$ is the vector of $m$ ones. For applications of blockers and anti-blockers, we refer, for example, to [14].

## 3. Exact blockers, exact anti-blockers, and $P_4$-free graphs

A blocker $\mathcal{B} = \mathcal{B}(\mathcal{H})$ (respectively, anti-blocker $\mathcal{A} = \mathcal{A}(\mathcal{H})$) is called *exact* if every minimal transversal $B \in \mathcal{B}$ (respectively, maximal anti-blocking set $A \in \mathcal{A}$) and each edge of $H \in \mathcal{H}$ have exactly one vertex in common, that is, if $|B \cap H| = 1$ (respectively, $|A \cap H| = 1$) for all $H \in \mathcal{H}$. Equivalently, in terms of the incidence matrix $C = C(\mathcal{H})$, a blocker (respectively, anti-blocker) is exact if and only if $Cx \equiv e$ whenever $x$ is a support-minimal (respectively, support-maximal) solution to $Cx \geq e$ (respectively, $Cx \leq e$).

Graph $P_4$ is defined on four vertices $\{v_1, v_2, v_3, v_4\}$ by three edges $P_4 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$.

It is self-complementary, that is, the complementary graph $\overline{P}_4 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$ is obviously isomorphic to $P_4$. Standardly, a graph $G$ is called $P_4$-free if it contains no induced $P_4$.

A hypergraph $\mathcal{H}$ will be called *B-exact (respectively, A-exact)* if its blocker $\mathcal{B}(\mathcal{H})$ (respectively, anti-blocker $\mathcal{A}(\mathcal{H})$) is exact. The $B$-exact hypergraphs are completely characterized by the following statement.

**Theorem 1** (*[18], See also [22,16,19]*). *The next five properties of a hypergraph $\mathcal{H}$ are equivalent:*

(i) *the blocker $\mathcal{B} = \mathcal{B}(\mathcal{H})$ to $\mathcal{H}$ is exact, that is, $\mathcal{H}$ is B-exact;*
(ii) *hypergraph $\mathcal{H}$ is completely clique-maximal and its co-occurrence graph $G(\mathcal{H})$ is $P_4$-free;*
(iii) *$|B \cap H| = 1$ for all $B \in \mathcal{B}(\mathcal{H})$ and $H \in \mathcal{H}$;*
(iv) *the co-occurrence graphs $G(\mathcal{H})$ and $G(\mathcal{B}(\mathcal{H}))$ are edge-disjoint;*
(v) *the co-occurrence graphs $G(\mathcal{H})$ and $G(\mathcal{B}(\mathcal{H}))$ are complementary.* □

**Remark 1.** It is also shown in [18,20,22,19,16] that $P_4$-free graphs are in one-to-one correspondence with the so-called read-once Boolean functions. A simple recognition algorithm for the latter was suggested in [17], see also [16]. Given a DNF $f$ of $n$ variables, this algorithm can verify whether $f$ is read-once and produces a (unique) read-once formula, when the answer is positive, in time $O(n|f|)$.

**Remark 2.** To show that complete clique-maximality is essential in (ii) let us consider the hypergraphs $\mathcal{H}_1$ and $\mathcal{H}_5$ from Example 1. They both are clique-maximal, but not completely, and none of them is $B$-exact, although their co-occurrence graphs are $P_4$-free: $G(\mathcal{H}_1) = K_3$ and $G(\mathcal{H}_5)$ is the complete 3-partite $2 \times 2 \times 2$ graph. Hypergraph $\mathcal{H}_4$ is completely clique-maximal but not $B$-exact, since $G(\mathcal{H}_4)$ contains a $P_4$.

Moreover, the clique hypergraphs $\mathcal{H}_C(G)$ of a $P_4$-free graph $G$ is not only $B$-exact but also $A$-exact.

Indeed, as we already know, if $\mathcal{H} = \mathcal{H}_C(G)$ then $\mathcal{A} = \mathcal{H}_S(G)$ is the anti-blocker of $\mathcal{H}$. Furthermore, if $G$ is a $P_4$-free graph then this anti-blocker is exact, by (iii). Thus, both the anti-blocker $\mathcal{A}(\mathcal{H})$ and blocker $\mathcal{B}(\mathcal{H})$ are exact whenever $\mathcal{H}$ satisfies (ii). By Theorem 1, (ii) is also *necessary* for $B$-exactness. Yet, not for $A$-exactness. In the next section, we will show that each $A$-exact hypergraph is clique-maximal but it might be not completely clique-maximal and its co-occurrence graph might contain an induced $P_4$.

Some necessary and some sufficient conditions for $A$-exactness will be also given in the next section.

## 4. On *A*-exact and clique-maximal hypergraphs

Clique-maximality is a necessary condition for $A$-exactness.

**Proposition 1.** *A hypergraph $\mathcal{H}$ is clique-maximal whenever it is A-exact.*

**Proof.** Let us assume indirectly that $\mathcal{H}$ is not clique-maximal; in other words, it has an edge $H_0 \in \mathcal{H}$ and its co-occurrence graph $G = G(\mathcal{H})$ has a (maximal) clique $C_0$ such that $H_0 \subset C_0$ and containment is strict, i.e., there is a vertex $v \in C_0 \setminus H_0$. Let $S_0$ be a maximal stable set in $G$ that contains $v$. Then obviously, $S_0$ is anti-blocking ($|S_0 \cap H| \leq 1$ for any $H \in \mathcal{H}$) and $S_0 \cap H_0 = \emptyset$. Thus, $\mathcal{H}$ is *not* $A$-exact. □

We call $G$ a *CIS graph* (or say that $G$ has the CIS property) if $C \cap S \neq \emptyset$ for every maximal clique $C$ and every maximal stable set $S$ in $G$. Each $P_4$-free graph is a CIS graph, yet, there are many others; see Section 5 and also [1] for more details. The following condition is sufficient for $A$-exactness.

**Proposition 2.** *A hypergraph $\mathcal{H}$ is A-exact whenever it is clique-maximal and its co-occurrence graph $G(\mathcal{H})$ is a CIS graph.*

**Proof.** As we already know, a maximal anti-blocking set $S$ of $\mathcal{H}$ is a maximal stable set of $G(\mathcal{H})$. Hence, $H \cap S \neq \emptyset$, since $H$ is clique-maximal and $G(\mathcal{H})$ is a CIS graph. □

However, $\mathcal{H}$ might be $A$-exact when $G(\mathcal{H})$ is not a CIS graph.

**Example 2.** Let us recall hypergraph $\mathcal{H}_3 = \{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_1)\}$ from Example 1. It is easy to verify that the co-occurrence graph $G(\mathcal{H}_3)$ is not a CIS graph, since $C \cap S = \emptyset$ for $C = \{(v_1, v_3, v_5)\}$ and $S = \{(v_2, v_4, v_6)\}$. Yet, the anti-blocker $\mathcal{A}(\mathcal{H}_3) = \{(v_1, v_4), (v_2, v_5), (v_3, v_6), (v_2, v_4, v_6)\}$ is exact.

Let us notice that the hypergraph $\mathcal{A}(\mathcal{H}_3)$ is not $A$-exact, although it is the exact anti-blocker to $\mathcal{H}_3$. Indeed, $S = (v_2, v_4, v_6) \in \mathcal{A}(\mathcal{H}_3)$, while it is easy to check that $C = (v_1, v_3, v_5) \in \mathcal{A}(\mathcal{A}(\mathcal{H}_3))$.

Furthermore, for the same reason, the completely clique-maximal hypergraph $\mathcal{H}_4 = \mathcal{H}_3 \cup \{(v_1, v_3, v_5)\}$ is not $A$-exact, either.

## 5. Main properties of CIS graphs

By definition, CIS graphs are closed under complementation.

It is also not difficult to show that they are *exactly closed* under substitution [1]. In other words, let notation $G = G'(v \to G'')$ mean that graph $G$ is obtained from graph $G'$ by substituting graph $G''$, as a module, for a fixed vertex $v$ in $G'$; then, $G$ is a CIS graph if and only if $G'$ and $G''$ are CIS graphs.

However, the family of CIS graphs is not hereditary. For example, let us consider the bull (or $A$-graph) $G = (V, E)$ defined by $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_2, v_5), (v_3, v_5)\}$. It is easy to verify that $G$ is a CIS graph, yet, it contains an induced $P_4$, which is not a CIS graph.

For this reason, CIS graphs cannot be characterized in terms of forbidden subgraphs. In fact, every graph $G'$ is an induced subgraph of a CIS graph $G$. Given $G'$, to get $G$ it is sufficient to add a simplicial vertex to each maximal clique of $G'$ (which does not have one already in $G'$). Let us note, however, that $G$ might be exponential in the size of $G'$. See [1] for more details.

Perhaps, for the same reason, no efficient characterization or recognition algorithm for CIS graphs is known. Yet, some necessary but not sufficient and sufficient but not necessary conditions are known.

For an integer $k \geq 2$, we define a *k-comb* $G_k$ as a graph on $2k$ vertices $\{v_1, \dots, v_k; v'_1, \dots, v'_k\}$ with $k(k+1)/2$ edges which form the clique on $\{v_1, \dots, v_k\}$ and perfect matching $(v_i, v'_i)$, $i \in [k] = \{1, \dots, k\}$.

The complementary graph $\overline{G}_k$ is called a $k$-anti-comb. Obviously, 2-comb, 2-anti-comb, and $P_4$ are three isomorphic graphs. It is easy to see that a $k$-comb contains $k$ induced $(k-1)$-combs for each $k \geq 3$.

It is also clear that $k$-combs $G_k$ and $k$-anti-combs $\overline{G}_k$ are not CIS graphs. Indeed, two disjoint sets $\{v_1, \dots, v_k\}$ and $\{v'_1, \dots, v'_k\}$ induce a maximal clique and maximal stable set in $G_k$, and vice versa in $\overline{G}_k$.

In the 1980s, Claude Berge noticed that in a CIS graph $G$ every induced $P_4$ must be contained in an induced bull-graph; see [27]. More generally, for each $k \geq 2$, in a CIS graph $G$, every induced $k$-comb $G_k$ (respectively, $k$-anti-comb $\overline{G}_k$) must be settled, that is, $G$ must contain a vertex $v_0$ adjacent to every $v_i$ and not adjacent to every $v'_i$ for all $i \in [k] = \{1, \dots, k\}$ (respectively, vice versa) [1]. Berge's necessary conditions correspond to the case $k = 2$. However, even for all $k \geq 2$, the above conditions do not imply the CIS property. The corresponding example was constructed by Ron Holzman in 1994; see [1].

By Theorem 1, $G$ is a CIS graph whenever it is $P_4$-free. In this case, $G$ contains no induced combs and anti-combs. In fact, the following relaxation still implies the CIS property.

**Theorem 2.** *Graph $G$ is a CIS graph whenever it contains no induced 3-combs and 3-anti-combs and every induced 2-comb is settled in $G$.*

This statement was conjectured in the early 1990s by Vasek Chvatal. His RUTCOR student Wenan Zang published first partial results in 1995 [27]. Finally, Theorem 2 was proved by Deng et al. [10,11], and independently in [1].

Graph $G$ is called an *almost CIS* graph if every its maximal clique $C$ and maximal stable set $S$ intersect, except a unique pair. In contrast to CIS graphs, the family of almost CIS graphs admits a simple (although non-trivial) characterization.

**Theorem 3.** *Graph $G$ is an almost CIS graph if and only if $G$ is a split graph with a unique split-partition.*

This statement was conjectured in [1]. First partial results were obtained in [6]. Recently, Theorem 3 was proved by Wu et al. [26].

## 6. On completely $\ell$-clique-maximal hypergraphs and $(\ell, \ell')$-CIS graphs

A Sperner hypergraph $\mathcal{H} \subseteq 2^V$ will be called *completely $\ell$-clique-maximal* if in its co-occurrence graph $G = G(\mathcal{H})$ every clique of cardinality at most $\ell$ is contained by an edge $H \in \mathcal{H}$.

First, without any loss of generality, we can assume that $2 \leq \ell \leq \omega$, where $\omega = \omega(G)$ is the so-called *clique number* of graph $G$, that is, the number of vertices of a maximum clique of $G$.

Indeed, every hypergraph is completely 2-clique-maximal, just by definition of the co-occurrence graph.

Furthermore, if $\mathcal{H}$ is a completely $\omega$-clique-maximal hypergraph then it is also completely $\ell$-clique-maximal for any $\ell$. In fact, $\mathcal{H}$ is completely $\omega$-clique-maximal if and only if it is completely clique-maximal.

**Remark 3.** Yet, let us notice that a completely $\ell$-clique-maximal hypergraph might be not clique-maximal when $\ell < \omega$. In general, if a hypergraph is completely $\ell$-clique-maximal then obviously it is completely $\ell'$-clique-maximality whenever $\ell \geq \ell'$. Let us note also that $\omega = \omega(G) \leq |V(G)| = n$ for every graph $G$.

Given integer $\ell$ and $\ell'$, a graph $G = (V, E)$ will be called $(\ell, \ell')$-*CIS graph* if there exist completely $\ell$- and $\ell'$-clique-maximal hypergraphs $\mathcal{H}$ and $\mathcal{H}'$ whose co-occurrence graphs are $G$ and $\overline{G}$ respectively, that is, $G(\mathcal{H}) = G$, $G(\mathcal{H}') = \overline{G}$, and whose edges pairwise intersect, that is, $H \cap H' \neq \emptyset$ for all $H \in \mathcal{H}, H' \in \mathcal{H}'$.

Again, without loss of generality, we assume that $2 \leq \ell \leq \omega(G)$ and $2 \leq \ell' \leq \omega(\overline{G}) = \alpha(G)$, where $\alpha = \alpha(G)$ is the so-called *stability number* of graph $G$. Moreover, the following statements hold.

**Proposition 3.** *Hypergraphs $\mathcal{H}$ and $\mathcal{H}'$ are clique-maximal for every $(\ell, \ell')$-CIS graph $G$.*

*If $\ell' \geq \alpha$ then hypergraph $\mathcal{H}'$ is completely clique-maximal, while hypergraph $\mathcal{H}$ is A-exact.*

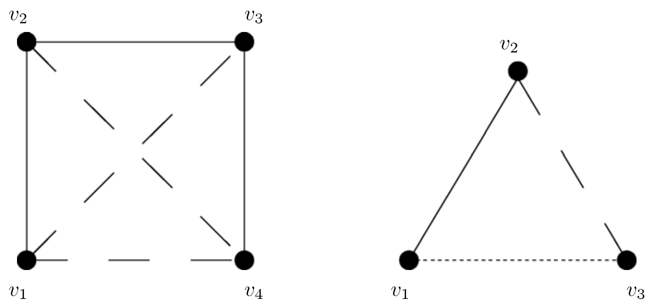


**Fig. 1.** 2- and 3-colored $d$-graphs $\Pi$ and $\Delta$.

**Proof.** The first claim is proved like Proposition 1, while the last two are straightforward. $\square$

Yet, $\mathcal{H}$ might be not completely clique-maximal and $\mathcal{H}'$ not $A$-exact even when $\ell' \geq \alpha$; see Example 2.

Finally, the above definitions easily result in the following characterization of $A$-exactness.

**Theorem 4.** *Let $\mathcal{H}$ be a hypergraph and $G = G(\mathcal{H})$ be its co-occurrence graph. Then $\mathcal{H}$ is not $A$-exact unless it is clique-maximal and $G$ is a $(2, \alpha(G))$-CIS graph. When both conditions hold then $\mathcal{H}$ is $A$-exact if and only if every its edge $H \in \mathcal{H}$ and every maximal stable set $S$ of $G$ intersect, $H \cap S \neq \emptyset$.* $\square$

By definition of the $(\ell, \ell')$-CIS property, such an $A$-exact hypergraph $\mathcal{H}$ exists for every given $(2, \alpha(G))$-CIS graph $G$. Let us also notice that Theorem 4 strengthens Propositions 1 and 2.

In Section 9 we will extend the above $(\ell, \ell')$-CIS property from graphs to $d$-graphs.

In the next section, we extend the standard CIS property from graphs to $d$-graphs.

## 7. CIS *d*-graphs

A $d$-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ is a complete graph on the vertex-set $V = \{v_1, \ldots, v_n\}$ whose $\binom{n}{2}$ edges are partitioned into $d$ subsets (colored by $d$ colors) some of which might be empty. We say that $\mathcal{G}$ is $\ell$-colored if $\ell$ is the number of its non-empty chromatic components $E_i \neq \emptyset$ for $i \in [d] = \{1, \ldots, d\}$.

Obviously, $\ell = 0$ if and only if $\mathcal{G}$ consists of a unique vertex, $|V| = 1$. Such $d$-graph is called *trivial*.

In case $d = 2$ a $d$-graph is just a graph, or more precisely, a pair that consists of a graph and its complement. Thus, $d$-graphs can be viewed as a generalization of graphs.

Given a $d$-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$, let $G_i = (V, E_i)$ denote its $i$th chromatic component, that is, the graph on the vertex-set $V$ with the edge-set $E_i$; furthermore, let $S_i \subseteq V$ be a *maximal* stable set in $G_i$, where $i \in [d]$; finally, let $\mathcal{S} = \{S_i \mid i \in [d]\}$ be a collection of $d$ such sets and let $S = \cap_{i=1}^{d} S_i$.

Obviously, $|S| \leq 1$ for every collection $\mathcal{S}$, since $v, v' \in S$ implies that edge $(v, v')$ has no color in $\mathcal{G}$.

We call $\mathcal{G}$ a *CIS d-graph*, or say that it has CIS $d$-property, if $S \neq \emptyset$ for each collection $\mathcal{S}$ defined above.

It is not difficult to verify that the family of CIS $d$-graphs is exactly closed under substitution [1,19]. More precisely, let $\mathcal{G}'$ and $\mathcal{G}''$ be two vertex-disjoint $d$-graphs and let $\mathcal{G} = \mathcal{G}'(v \to \mathcal{G}'')$ denote the $d$-graph obtained by substituting $\mathcal{G}''$ for a vertex $v$ in $\mathcal{G}'$. Then, $\mathcal{G}$ has the CIS property if and only if both $\mathcal{G}'$ and $\mathcal{G}''$ have it. Let us also recall that the CIS property is still not hereditary for $d = 2$.

## 8. On remarkable properties of *d*-graphs $\Pi$ and $\Delta$

### 8.1. Definition

Two $d$-graphs $\Pi$ and $\Delta$ given in Fig. 1 play an important role:

$\Pi$ is defined for any $d \geq 2$ by $V = \{v_1, v_2, v_3, v_4\}$;

$E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$, $E_2 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$, and $E_i = \emptyset$ whenever $i > 2$;

$\Delta$ is defined for any $d \geq 3$ by $V = \{v_1, v_2, v_3\}$,

$E_1 = \{(v_1, v_2)\}$, $E_2 = \{(v_2, v_3)\}$, $E_3 = \{(v_3, v_1)\}$, and $E_i = \emptyset$ whenever $i > 3$.

Clearly, $\Pi$ and $\Delta$ are respectively 2- and 3-colored $d$-graphs; both non-empty chromatic components of $\Pi$ are isomorphic to $P_4$ and $\Delta$ is a three-colored triangle.

Both $d$-graphs $\Pi$ and $\Delta$ were introduced in 1967 by Gallai in his seminal paper [13]; $\Delta$-free $d$-graphs are frequently referred to as Gallai's graphs; we will call them Gallai's $d$-graphs, which is more accurate.

It is easy to verify that the class of Gallai's $d$-graphs is exactly closed under substitution [1,19] and hereditary, just by definition. (Recall that CIS $d$-graphs have only the former but not the latter property.)

### 8.2. Minimal and locally minimal complementary connected d-graphs

A $d$-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ is called *complementary connected* (CC) if the complement $G(V, \overline{E_i})$ to its $i$th chromatic component $G(V, E_i)$ is connected on $V$ for all $i \in [d]$.

**Lemma 1.** *For an arbitrary d-graph $\mathcal{G}$ there is at most one $i \in [d]$ such that $G(V, \overline{E_i})$ is not connected.*

**Proof.** If $G(V, \overline{E_i})$ is not connected on $V$ then, obviously, $G(V, E_i)$ is connected and, hence, $G(V, \overline{E_j})$ is connected too, whenever $j$ is distinct from $i$, since in this case, $G(V, E_i)$ is a subgraph of $G(V, \overline{E_j})$. □

Obviously, $\Pi$ and $\Delta$ are minimal CC $d$-graphs, that is, they are CC, while all their proper sub-$d$-graphs are not. (By convention, we assume that the trivial, single-vertex, $d$-graph is not CC.) Moreover, except for $\Pi$ and $\Delta$, there are no other minimal CC $d$-graphs.

**Theorem 5.** *Every CC d-graph contains a $\Pi$ or $\Delta$ as a subgraph.*

This result was proven in [20]; see also [5,19].

By this theorem and Lemma 1, for every $\Pi$- and $\Delta$-free $d$-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ there is a unique $i \in [d]$ such that $\overline{G_i} = G(V, \overline{E_i})$ is not connected on $V$. Let us split $\overline{G_i}$ into connected components and partition $V$ accordingly. Since the corresponding sub-$d$-graphs are still $\Pi$- and $\Delta$-free, we can proceed with such partitioning until we obtain finally a unique canonical decomposition of $\mathcal{G}$ [20,19].

In case $d = 2$, this is the well-known modular decomposition of the $P_4$-free graphs.

As a corollary, we obtain a one-to-one correspondence between the $\Pi$- and $\Delta$-free $d$-graphs and extensive $d$-person game forms; see [20,19] for more details. In [5], Theorem 5 was extended as follows.

**Theorem 6.** *There are exactly two locally minimal CC d-graphs: $\Pi$ and $\Delta$; that is, any other CC d-graph $\mathcal{G}$ contains a vertex $v \in V$ such that the sub-d-graph $\mathcal{G}[V \setminus \{v\}]$ is still CC.*

### 8.3. Minimal and locally minimal non-CIS d-graphs

It is also easily seen that $\Pi$ and $\Delta$ are minimal non-CIS $d$-graphs, that is, the CIS property does not hold for $\Pi$ and $\Delta$ but it holds for all their proper sub-$d$-graphs. Moreover, except for $\Pi$ and $\Delta$, there are no other minimal non-CIS $d$-graphs. Let us notice that the trivial, single-vertex, $d$-graph has the CIS property.

**Theorem 7.** *Every non-CIS d-graph contains a $\Pi$ or $\Delta$, or in other words, all $\Pi$- and $\Delta$-free d-graphs have the CIS d-property.*

In [20,19], this result was derived from the above canonical decomposition of the $\Pi$- and $\Delta$-free $d$-graphs.

We will give a shorter proof (of a stronger statement) in the next section.

In [2], Theorem 7 was also strengthened, in a different way, as follows:

**Theorem 8.** *The only locally minimal non-CIS d-graphs are $\Pi$ and $\Delta$, that is, any other non-CIS d-graph $\mathcal{G}$ contains a vertex $v \in V$ such that the sub-d-graph $\mathcal{G}[V \setminus \{v\}]$ is still non-CIS.*

**Remark 4.** Thus, $\Pi$ and $\Delta$ are the only minimal and, moreover, they are the only locally minimal elements of the following two classes: CC and non-CIS $d$-graphs. It was shown in [2] that these two classes are in general position, that is, one does not contain the other; they intersect, since both contain $\Pi$ and $\Delta$.

### 8.4. Another generalization of Theorem 7 and its proof

Theorem 7 follows from Theorem 8 but the proof of the latter in [2] is pretty long. Also, Theorem 7 can be derived from Theorem 5 and resulting from it canonical decomposition of the $\Pi$- and $\Delta$-free $d$-graphs. Yet, this plan, realized in [20,19], is complicated too. Here we suggest one more generalization of Theorem 7 (see Theorem 9) and a relatively short proof of it obtained recently by Endre Boros and the author.

In [3], a cycle of a $d$-graph is called *colorful* if all its edges have pairwise distinct colors. Obviously, this concept can be extended to the paths, as well.

**Lemma 2.** *A Gallai d-graph has no colorful cycles, or in other words, a d-graph with a colorful cycle has a colorful triangle, that is, $\Delta$.*

This claim is instrumental in [3]. The induction on the number of edges of the cycle is obvious. □

By definition, a non-CIS $d$-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ has a collection $\mathcal{S} = \{S_i \subseteq V \mid i \in [d]\}$ whose intersection is empty, $S = \cap_{i=1}^{d} S_i = \emptyset$, where $S_i$ is a maximal independent set of the $i$th chromatic component $G_i = (V, E_i)$ of $\mathcal{G}$ for each $i \in [d] = \{1, \ldots, [d]\}$.

Let us choose a vertex $v^0 \in V$. It does not belong to a maximal independent set of $\mathcal{S}$, say, to $S_{i_1}$, since $S = \emptyset$. Then, there is a vertex $v_{i_1} \in S_{i_1}$ such that $(v^0, v_{i_1}) \in E_{i_1}$, since otherwise set $S_{i_1} \cup \{v^0\}$ would be independent in $G_{i_1}$, in contradiction with maximality of $S_{i_1}$. In its turn, $v_{i_1}$ does not belong to a maximal independent set, say, to $S_{i_2}$. Again by maximality, there is a $v_{i_2} \in S_{i_2}$ such that $(v_{i_1}, v_{i_2}) \in E_{i_2}$, etc. Since $d$-graph $\mathcal{G}$ is finite, this procedure will result in a cycle $C$ that consists of $k$

distinct vertices $v_{i_j} \in S_{i_j}$ and $k$ edges $(v_{i_{j-1}}, v_{i_j}) \in E_{i_j}$, where $j \in \{t, t+1, \dots, t+k-1\}$. Without any loss of generality, we can assume that $t = 1$ and $j \in [k] = \{1, \dots, k\}$. Standardly the indices are taken modulo $k$, that is, $v_{i_0} = v_{i_k}$. Let us notice that $C$ might contain $v_0$; in this case we assume, again without any loss of generality, that $v^0 = v_{i_0} = v_{i_k}$.

A cycle $C$ obtained in such a way will be called a *$\Pi\Delta$-cycle* in $\mathcal{G}$.

Let us generalize this concept slightly and extend it to all, CIS or non-CIS, $d$-graphs. To do so, we relax the above definition a bit assuming that $S_{i_j}$ is an (*not necessarily maximal*) independent set of $G_{i_j} = (V, E_{i_j})$, for $j \in [k]$, $i_j \in [d]$. In other words, a $\Pi\Delta$-cycle $C$ in $\mathcal{G}$ is defined by the following condition:

if edges $(v_{i_{r-1}}, v_{i_r})$ and $(v_{i_{s-1}}, v_{i_s})$ of $C$ are of the same color then $(v_{i_r}, v_{i_s})$ must be of a different color:

$$(v_{i_{r-1}}, v_{i_r}), (v_{i_{s-1}}, v_{i_s}) \in E_m \Rightarrow (v_{i_r}, v_{i_s}) \notin E_m \quad \forall r, s \in [k],\ m \in [d].$$

**Remark 5.** The order of vertices $v_{i_1}, \dots, v_{i_k}$ cannot be reversed. In fact, $C$ is a directed cycle.

The above arguments result in the following statement.

**Lemma 3.** *Every non-CIS $d$-graph contains a $\Pi\Delta$-cycle.* □

**Lemma 4.** *Furthermore, $\Pi$ and $\Delta$ contain Hamiltonian $\Pi\Delta$-cycles.*

**Proof.** Indeed, in $\Pi$ such a cycle $C$ is specified by the sequence of vertices $\{v_1, v_3, v_4, v_2\}$, in other words, $i_0 = i_4 = 2, i_1 = 1, i_2 = 3, i_3 = 4$; see Fig. 1. Thus, colors in $C$ alternate: $v_1, v_4 \in S_1,\ v_2, v_3 \in S_2$, while in $\Delta$ all colors are distinct: $S_{i_j} = j$ for $j \in \{1, 2, 3\}$; see Fig. 1. □

In particular, the above lemma and Theorem 5 imply that every CC $d$-graph contains a $\Pi\Delta$-cycle.

**Remark 6.** Let us also mention that CIS $d$-graphs can contain $\Pi\Delta$-cycles as well, already for $d = 2$.

For example, 2-graph $\Pi$, which has a $\Pi\Delta$-cycle $C$, can be extended to a bull-graph (also called A-graph), which has the CIS property but still contains $C$.

In contrast, the absence of the $\Pi\Delta$-cycles is a characteristic property of the $\Pi$- and $\Delta$-free $d$-graphs.

**Theorem 9.** *A $d$-graph contains a $\Pi$ or $\Delta$ if and only if it contains a $\Pi\Delta$-cycle.*

Obviously, this statement implies Theorem 7, since each non-CIS $d$-graph contains a $\Pi\Delta$-cycle.

**Proof of the theorem.** The "only if part" follows from Lemma 4. To prove the "if part", let us assume indirectly that a $\Pi$- and $\Delta$-free $d$-graph $\mathcal{G} = (V; E_1, \dots, E_d)$ contains a $\Pi\Delta$-cycle $C$. Without loss of generality, we can also assume that $C$ is a *shortest* $\Pi\Delta$-cycle in *all* $\Pi$- and $\Delta$-free $d$-graphs.

**Lemma 5.** *An edge $(v_{i_{j-1}}, v_{i_j})$ and diagonal $(v_{i_r}, v_{i_j})$ in $C$ are colored differently, while $(v_{i_r}, v_{i_{j-1}})$ is colored with one of these two colors, that is,*

$$(v_{i_{j-1}}, v_{i_j}) \in E_m \quad \textit{and} \quad (v_{i_r}, v_{i_j}) \in E_\ell \Rightarrow m \neq \ell \textit{ and } (v_{i_r}, v_{i_{j-1}}) \in E_m \cup E_\ell.$$

**Proof.** Indeed, if $m = \ell$ then a $\Pi\Delta$-cycle shorter than $C$ can be constructed in $\mathcal{G}$ in an obvious way.

Furthermore, if $(v_{i_r}, v_{i_{j-1}}) \notin E_m \cup E_\ell$ then three vertices $v_{i_{j-1}}, v_{i_j}$ and $v_{i_r}$ form a $\Delta$. □

By Lemma 2, $C$ contains a $\Delta$ whenever $C$ is colorful, that is, all its edges are colored with distinct colors.

We will assume that $C$ contains two edges of the same color and get a contradiction.

**Lemma 6.** *Any two successive edges of $C$ are colored with distinct colors.*

**Proof.** It follows if we just set $i_r = i_{s-1}$ in the definition of a $\Pi\Delta$-cycle. □

**Lemma 7.** *Any two edges of $C$ at distance 1 are colored with distinct colors, that is,*

$$(v_{i_{j-1}}, v_{i_j}) \in E_m, \qquad (v_{i_{j+1}}, v_{i_{j+2}}) \in E_{m'} \Rightarrow m \neq m'.$$

**Proof.** Let us assume indirectly that $m = m'$ and let $(v_{i_j}, v_{i_{j+1}}) \in E_\ell$. By these assumptions and Lemma 5,

$$(v_{i_{j-1}}, v_{i_j}), (v_{i_{j-1}}, v_{i_{j+1}}), (v_{i_{j+1}}, v_{i_{j+2}}) \in E_m; \qquad (v_{i_j}, v_{i_{j+1}}), (v_{i_j}, v_{i_{j+2}}), (v_{i_{j-1}}, v_{i_{j+2}}) \in E_\ell.$$

Since $m \neq \ell$, we conclude that the considered four successive vertices result in a $\Pi$ and contradiction. □

The above three lemmas immediately result in the following one.

**Lemma 8.** *For any four successive vertices $v_{i_0}, v_{i_1}, v_{i_2}, v_{i_3}$ of $C$, the corresponding three successive edges $(v_{i_0}, v_{i_1}), (v_{i_1}, v_{i_2})$, $(v_{i_2}, v_{i_3})$ are colored by three distinct colors $i_1, i_2, i_3$, respectively.* □

**Lemma 9.** *Moreover, edges $(v_{i_0}, v_{i_1}), (v_{i_0}, v_{i_2}), (v_{i_0}, v_{i_3})$ are of color $i_1$, edges $(v_{i_1}, v_{i_2}), (v_{i_1}, v_{i_3})$ are of color $i_2$, and $(v_{i_2}, v_{i_3})$ is of color $i_3$.*

**Proof.** Indeed, it is obvious that otherwise a $\Delta$ would appear. □

Then, one can easily proceed with induction and generalize the last two claims as follows.

**Lemma 10.** *For any $k+1$ successive vertices $v_{i_0}, v_{i_1}, \ldots, v_{i_k}$ of $C$, the corresponding $k$ successive edges $(v_{i_0}, v_{i_1}), \ldots, (v_{i_{k-1}}, v_{i_k})$ are colored by colors $j_1, \ldots, j_k$, respectively. These $k$ colors are pairwise distinct. Moreover, edge $(v_{i_j}, v_{i_{j'}})$ is of color $i_{j+1}$ for all $j$ and $j'$ such that* $0 \le j < j' \le k+1$. $\square$

Although such a colorful path can be arbitrarily long, yet obviously, it can never form a cycle.
The obtained contradiction proves Theorem 9.

*8.5. $\Delta$-conjecture*

As we know, CIS $d$-graphs may contain a $\Pi$, yet, it seems that they cannot contain a $\Delta$.
*$\Delta$-conjecture* ([20]; page 71, Remark after Claim 17).
*Each CIS $d$-graph is a Gallai $d$-graph*; *or in other words, no CIS $d$-graph contains a $\Delta$.*
Several partial results in this direction are obtained in [1]; in particular, $\Delta$-conjecture for an arbitrary $d$ is reduced to the case $d = 3$.

It is also shown in [1] (Sections 1.6, 1.7, and 4) that, modulo $\Delta$-conjecture, the problem of characterizing the CIS $d$-graphs can be reduced to the case $d = 2$, that is, to characterization of the CIS graphs. Let us remark, however, that case $d = 2$ is still very difficult [10,11,1]. The above reduction is based on the general concept of modular decomposition applied to the $\Delta$-free $d$-graphs [1–3,7–9,12,13,19,21,23,24].

## 9. On $\ell$-CIS $d$-graphs

Let us extend the concept of CIS $d$-graph as follows. Let $\ell = (\ell_1, \ldots, \ell_d)$ be a positive integer vector.

A $d$-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ will be called an $\ell$-CIS $d$-graph if for each $i \in [d] = \{1, \ldots, d\}$ there is a completely $\ell_i$-clique-maximal hypergraph $\mathcal{H}_i$ whose co-occurrence graph is $\overline{G}_i$ (hence, without loss of generality, we can assume that $\ell_i \ge 2$) and such that $\cap_{i=1}^{d} H_i \neq \emptyset$ for every $d$-uple $\{H_i \in \mathcal{H}_i \mid i \in [d]\}$.

Obviously, the $\ell$-CIS $d$-graphs turn into the standard CIS $d$-graphs when $\ell = (n, \ldots, n)$ and $n = |V|$.

In this case, all $\mathcal{H}_i$ are completely clique-maximal hypergraphs. In general, it is not difficult to demonstrate (just by copying the proof of Proposition 1) that all $\mathcal{H}_i$ are clique-maximal hypergraphs whenever $\mathcal{G}$ is an $\ell$-CIS $d$-graph. Furthermore, copying case analysis from [1], it is also easy to verify that

*$\ell$-CIS $d$-graphs are exactly closed under substitution.*

Hence, the $\Delta$-free (Gallai) $\ell$-CIS $d$-graphs can be reduced to $\ell$-CIS 2-graphs (that is, graphs) by modular decomposition, in accordance with [1,19]; see also [3,7–9].

However, $\Delta$-conjecture does not extend to the case $d = 3$ and $\ell = (2, 2, 2)$ (or even $\ell = (2, 2, 5)$). The next example was constructed by Andrey Gol'berg (1954–1985) in 1984.

**Example 3.** Let us consider the 3-graph $\mathcal{G}$ on nine vertices $V = \{v_0, v_1, \ldots, v_8\}$ in Fig. 2, where solid (dotted) lines are colored by color 3 (respectively, 2), and each edge between $\{v_1, v_2, v_3, v_4\}$ and $\{v_5, v_6, v_7, v_8\}$ is of color 1. It is easy to verify that $\mathcal{G}$ contains eight $\Delta$s induced by the vertex-sets

$$(v_0, v_1, v_6), (v_0, v_1, v_7), (v_0, v_4, v_6), (v_0, v_4, v_7), (v_0, v_2, v_5), (v_0, v_2, v_8), (v_0, v_3, v_5), (v_0, v_3, v_8).$$

Let us consider the following three hypergraphs: $\mathcal{H}_1 = \{(v_0, v_1, v_2, v_3, v_4), (v_0, v_5, v_6, v_7, v_8)\}$;

$$\mathcal{H}_2 = \{(v_0, v_2, v_3, v_6, v_7), (v_1, v_2, v_5, v_6), (v_1, v_2, v_7, v_8), (v_3, v_4, v_5, v_6), (v_3, v_4, v_7, v_8)\};$$

$$\mathcal{H}_3 = \{(v_0, v_1, v_4, v_5, v_8), (v_1, v_3, v_5, v_7), (v_1, v_3, v_6, v_8), (v_2, v_4, v_5, v_7), (v_2, v_4, v_6, v_8)\}.$$

It is also easy to verify that:

- (a) their co-occurrence graphs are $\overline{G}_1$, $\overline{G}_2$, and $\overline{G}_3$, respectively;
- (b) $\mathcal{H}_1$ is completely clique-maximal, while $\mathcal{H}_2$ and $\mathcal{H}_3$ are not; more precisely, they are completely 2-clique-maximal but not completely 3-clique-maximal; indeed, set $\{v_1, v_4, v_6, v_8\}$ is a 4-clique of $\overline{G}_3$, every its 2-subset is contained in an edge of $\mathcal{H}_3$, while the 3-subset $\{v_1, v_4, v_6\}$ is already not;
- (c) $H_1 \cap H_2 \cap H_3 \neq \emptyset$ (in fact, $|H_1 \cap H_2 \cap H_3| = 1$) for every $H_1 \in \mathcal{H}_1, H_2 \in \mathcal{H}_2, H_3 \in \mathcal{H}_3$.

The corresponding $2 \times 5 \times 5$ intersection table is given below.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $v_4$ | $v_4$ | $v_2$ | $v_2$ | $v_2$ | $v_6$ | $v_8$ | $v_6$ | $v_6$ | $v_8$ |
| $v_4$ | $v_4$ | $v_2$ | $v_2$ | $v_2$ | $v_5$ | $v_7$ | $v_7$ | $v_5$ | $v_7$ |
| $v_4$ | $v_4$ | $v_0$ | $v_1$ | $v_1$ | $v_5$ | $v_8$ | $v_0$ | $v_5$ | $v_8$ |
| $v_3$ | $v_3$ | $v_3$ | $v_1$ | $v_1$ | $v_6$ | $v_8$ | $v_6$ | $v_6$ | $v_8$ |
| $v_3$ | $v_3$ | $v_3$ | $v_1$ | $v_1$ | $v_5$ | $v_7$ | $v_7$ | $v_5$ | $v_7$. |

This table represents a 3-dimensional box-partition with many interesting properties; see [25].

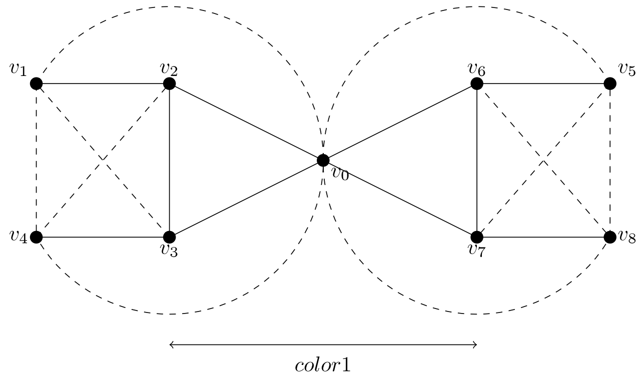


**Fig. 2.** A (2, 2, 5)-CIS 3-graph that contains eight $\Delta$s.

As we just mentioned, the hypergraphs $\mathcal{H}_2$ and $\mathcal{H}_3$ are clique-maximal but not completely clique-maximal. Their completely clique-maximal extensions are

$$\mathcal{H}_2' = \mathcal{H}_2 \cup \{(v_1, v_2, v_6, v_7), (v_3, v_4, v_6, v_7), (v_2, v_3, v_5, v_6), (v_2, v_3, v_7, v_8)\} \quad \text{and}$$
$$\mathcal{H}_3' = \mathcal{H}_3 \cup \{(v_1, v_4, v_5, v_7), (v_1, v_4, v_6, v_8), (v_1, v_3, v_5, v_8), (v_2, v_4, v_5, v_8)\}.$$

However, for the triplet $\mathcal{H}_1$, $\mathcal{H}_2'$, and $\mathcal{H}_3'$ the intersection property fails. For example,

$$\{v_0, v_5, v_6, v_7, v_8\} \cap \{v_1, v_2, v_6, v_7\} \cap \{v_1, v_3, v_5, v_8\} = \emptyset.$$

Hence, there is no contradiction with the "standard" $(n, n, n)$-CIS $\Delta$-conjecture.

Thus, a (2, 2, 5)-CIS 3-graph can contain a $\Delta$, while an $(n, n, n)$-one cannot, if $\Delta$-conjecture holds.
More generally, one can ask for which $\ell$, if any, the $\ell$-CIS $d$-graphs contain no $\Delta$.

## 10. On Gallai *d*-graphs and complete, normal, and solid *d*-dimensional box-partitions

The obtained intersection table $g : \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3 \to \{v_0, v_1, \dots, v_8\}$ represents a box-partition of the total $2 \times 5 \times 5$ box $\mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3$ into nine boxes $\{v_0, v_1, \dots, v_8\}$. Let us notice that the first five boxes in this box-partition are *solid*, that is, the corresponding edges got successive numbers in the given edge-enumeration of hypergraphs $\mathcal{H}_1$. $\mathcal{H}_2$, and $\mathcal{H}_3$, while the last four boxes are *not* solid. It is easy to verify that there is no enumeration of edges in these three hypergraphs such that *all* boxes are solid.

Let us give more details. A collection of $d$ hypergraphs $\boldsymbol{\mathcal{H}} = \{\mathcal{H}_i \subseteq 2^V \mid i \in [d] = \{1, \dots, d\}\}$ defined on a common vertex-set $V$ will be called a CIS collection (or we will say that it has the CIS property) if

$$\left| \bigcap_{i=1}^{d} H_i \right| = 1 \quad \text{for all } d\text{-uples } \mathbf{H} = \{H_i \in \mathcal{H}_i \mid i \in [d]\}.$$

Without any loss of generality, we assume that each vertex $v \in V$ is realized as the edge-intersection $v = \cap_{i=1}^{d} H_i$ of such a $d$-uple $\mathbf{H}$; indeed, all other vertices can be just removed.

Let us consider a mapping $g$ that assigns the intersection-vertex $v = v(\mathbf{H}) = \cap_{i=1}^{d} H_i$ to every $d$-uple $\mathbf{H} = \{H_i \in \mathcal{H}_i \mid i \in [d]\}$ of a CIS collection $\mathcal{H}$ of hypergraphs.

Alternatively, this mapping $g$ can be interpreted as a *box-partition* in which every vertex $v \in V$ is a box.

To each such box-partition $g$ we will assign a $(d+1)$-graph $\mathcal{G} = \mathcal{G}(g) = (V, E_0, E_1, \dots, E_d)$ as follows.

For every two distinct vertices $v, v' \in V$, let us define a subset $s(v, v') \subseteq [d]$ by the condition:

$i \notin s(v, v')$ if and only if $v, v' \in H_i$ for an edge $H_i \in \mathcal{H}_i$.

Obviously, $s(v, v') = \emptyset$ means that $\mathcal{H}$ is not a CIS collection (and $g(\mathcal{H})$ is not a box-partition), since boxes $v$ and $v'$ intersect. Further, $|s(v, v')| = 1$, say, $s(v, v') = \{i\} \in [d]$ if and only if projections *of the interiors* of boxes $v$ and $v'$ in the direction $i$ intersect. In this case let $(v, v') \in E_i$ in $\mathcal{G}$. Finally, $|s(v, v')| > 1$ if and only if projections *of the interiors* of $v$ and $v'$ intersect in no direction $i \in [d]$. In this case let $(v, v') \in E_0$ in $\mathcal{G}$. By this rule, to each box-partition $g : \mathcal{H} \to V$ a $(d+1)$-graph $\mathcal{G}(g) = (V; E_0, E_1, \dots, E_d)$ is assigned.

**Proposition 4.** *This $(d+1)$-graph $\mathcal{G}(g)$ contains a $\Delta$ whenever $E_0 \neq \emptyset$.*

**Proof.** It will easily result from the following statement.

**Lemma 11.** *Between any two vertices $v$ and $v'$ of $\mathcal{G}(g)$ there is a $E_0$-free colorful path.*

**Proof.** Let us choose in the box-partition $g$ two arbitrary states $x$ and $x'$ from the boxes $v$ and $v'$, respectively, and consider a *Hamming path* $P$ between $x$ and $x'$. [By definition, each edge of $P$ reduces the number of distinct coordinates (the so-called Hamming distance) between the current state and $x'$ by 1.] Let us notice that several successive edges of $P$ may stay in one box, yet, $P$ cannot leave a box and then return to it. Let us consider all edges of $P$ that go from a box to a distinct one. Obviously, these edges define a colorful $E_0$-free path in $\mathcal{G}(g)$. □

To finish with Proposition 4, let us choose an edge $(v, v') \in E_0$ in $\mathcal{G}(g)$. This edge and a $E_0$-free colorful path between $v$ and $v'$ form a colorful cycle. Then, by Lemma 2, $\mathcal{G}(g)$ contains a $\Delta$. □

Furthermore, a box-partition $g = g(\mathcal{H})$ will be called:

- (i) *complete*, if $E_0 = \emptyset$, or in other words, if for each $v, v' \in V$ there is a direction $i \in [d]$ such that projections of the interiors of boxes $v$ and $v'$ in this direction intersect;
- (ii) *normal*, if $g$ is complete and all $d$ hypergraphs of $\boldsymbol{\mathcal{H}}$ are completely clique-maximal; or in other words, if for every direction $i \in [d]$ the following Helly property holds: projections, in the direction $i$, of the interiors of a family of boxes intersect whenever they are pairwise intersect;
- (iii) *Gallai's*, if $g$ is complete and the corresponding $d$-graph $\mathcal{G}(g)$ is $\Delta$-free;
- (iv) *solid*, if there is an enumeration of the edges in each of the $d$ hypergraphs of $\boldsymbol{\mathcal{H}}$ such that all boxes of the box-partition $g$ are solid.

As we know, the box-partition $g$ from Example 3 is complete but not normal, not solid, and not Gallai's.

Obviously, the $\Delta$-conjecture can be reformulated as follows: *any normal box-partition is Gallai's*.

Let us remark that every complete and solid box-partition is Gallai's, indeed.

**Theorem 10.** *If a box-partition $g$ is complete and solid then its $d$-graph $\mathcal{G}(g)$ contains no $\Delta$.* □

This statement was announced in [15], a proof first appeared in [25].

The result admits a natural geometric interpretation:

*no three solid boxes that induce a $\Delta$ can be extended to a complete solid box-partition.*

However, Example 3 shows that the similar statement fails if boxes may be not solid.

## Acknowledgements

The author is grateful to the anonymous referees for many helpful remarks and corrections and to Endre Boros for his contribution to the proof of Theorem 9. The author is also grateful to DIMACS, Center for Discrete Mathematics and Theoretical Computer Science, Graduate School of Information Science and Technology at University of Tokyo, Center for Algorithmic Game Theory at University of Aarhus, INRIA at Ecole Polytechnique, Department of Mathematics at University of Pierre and Marie Curie, Paris 6, and Max Planck Institute for Informatics at Saarbrücken for partial support.

## References

[1] D. Andrade, E. Boros, V. Gurvich, On graphs whose maximal cliques and stable sets intersect, RUTCOR Research Report 17-2006 and DIMACS Technical Report 2006-16, Rutgers University.
[2] D. Andrade, E. Boros, V. Gurvich, Not complementary connected and not CIS $d$-graphs form weakly monotone families, Discrete Math. 310 (5) (2010) 1089–1096. RUTCOR Research Report, 04-2009, Rutgers University.
[3] R.N. Ball, A. Pultr, P. Vojtěchovský, Colored graphs without colorful cycles, Combinatorica 27 (4) (2007) 407–427.
[4] C. Berge, Graphes et Hypergraphes, Dunod, Paris, 1970, English Translation, Graphs and Hypergraphs, North-Holland, Amsterdam, 1973.
[5] E. Boros, V. Gurvich, Vertex- and edge-minimal and locally minimal graphs, Discrete Math. 309 (12) (2009) 3853–3865. RUTCOR Research Report RRR-25-2007, Rutgers University.
[6] E. Boros, V. Gurvich, I. Zverovich, On split and almost CIS graphs, Australas. J. Combin. 43 (2009) 163–180.
[7] K. Cameron, J. Edmonds, Lambda composition, J. Graph Theory 26 (1997) 9–16.
[8] K. Cameron, J. Edmonds, L. Lovász, A note on perfect graphs, Period. Math. Hungar. 17 (3) (1986) 441–447.
[9] F.R.K. Chung, R.L. Graham, Edge-colored complete graphs with precisely colored subgraphs, Combinatorica 3 (1983) 315–324.
[10] X. Deng, G. Li, W. Zang, Proof of Chvátal's conjecture on maximal stable sets and maximal cliques in graphs, J. Combin. Theory Ser. B 91 (2) (2004) 301–325.
[11] X. Deng, G. Li, W. Zang, Corrigendum to: "Proof of Chvátal's conjecture on maximal stable sets and maximal cliques in graphs", J. Combin. Theory Ser. B 94 (2) (2005) 352–353.
[12] P. Erdős, M. Simonovits, V.T. Sós, Anti-Ramsey theorems, Colloq. Math. Soc. János Bolyai 10 (1973) 633–643.
[13] T. Gallai, Transitiv orientierbare graphen, Acta Math. Acad. Sci. Hungar. 18 (1–2) (1967) 25–66. Translated by F. Maffray and M. Preissmann, in: J.L.R Alfonsin and B.A. Reed (Eds.), Perfect Graphs (Chapter 3), John Wiley and Sons, 2002.
[14] G. Gasparyan, A. Sebo, M. Preissmann, Imperfect and nonideal clutters: a common approach, Combinatorica 23 (2) (2003) 283–302.
[15] A. Gol'berg, V. Gurvich, A problem in the theory of extremal graphs and its applications, Izv. Akad. Nauk SSSR Ser. Tekh. Kibernet. 1 (1988) 106–116. (in Russian); English transl. in Soviet J. of Computer and System Science (formerly Engineering Cybernetics) 3, 1990, 1–9.

[16] M.C. Golumbic, V. Gurvich, Read-once Boolean functions, in: Y. Crama and P.L. Hammer (Eds.), Boolean Functions: Theory, Algorithms, and Applications, 2011, Cambridge University Press (in press) (Chapter 12).

[17] M.C. Golumbic, A. Minz, U. Rotics, An improvement on the complexity of factoring read-once Boolean functions, DAM 156 (2008) 1633–1636.

[18] V. Gurvich, On repetition-free Boolean functions, Russian Math. Surveys 32 (1977) 183–184 (in Russian).

[19] V. Gurvich, Decomposing complete edge-chromatic graphs and hypergraphs. Revisited, Discrete Appl. Math. 157 (2009) 3069–3085. RUTCOR Research Report 29-2006 and DIMACS Technical Report 2007-04, Rutgers University.

[20] V. Gurvich, Applications of Boolean functions and networks in game theory, Ph.D. Thesis, Graduate School of Moscow Institute of Physics and Technology, Moscow, USSR, 1978 (in Russian); (Chapter 2, Nash solvability in pure strategies; Chapter 5, Repetition-free Boolean functions and normal forms of positional games).

[21] A. Gyárfás, G. Simonyi, Edge coloring of complete graphs without tricolored triangles, J. Graph Theory 46 (2004) 211–216.

[22] M. Karchmer, N. Linial, L. Newman, M. Saks, A. Wigderson, Combinatorial characterization of read-once formulae, Discrete Math. 114 (1993) 275–282.

[23] J. Körner, G. Simonyi, Graph pairs and their entropies: modularity problems, Combinatorica 20 (2000) 227–240.

[24] J. Körner, G. Simonyi, Z. Tuza, Perfect couples of graphs, Combinatorica 12 (1992) 179–192.

[25] Y. Liu, Combinatorial box-partitions, box packing, and their applications, Ph.D. Dissertation, RUTCOR, Rutgers University, October 2003.

[26] Y. Wu, W. Zang, C.-Q. Zhang, A characterization of almost CIS graphs, SIAM J. Discrete Math. 23 (2009) 749–753.

[27] W. Zang, Generalizations of Grillet's theorem on maximal stable sets and maximal cliques in graphs, Discrete Math. 143 (1995) 259–268.